RESEARCH DEPARTMENT

The effect of small variations in the coefficients of an equation upon its roots

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SUMMARY

Given a "new" algebraic equation which can be regarded as a variation of an "original" algebraic equation whose well-separated roots are known initially, the roots of the "new" equation are here derived explicitly in terms of Newton's well-known formula. The adaptation of this formula to cases where the roots of the "original" equation are not well separated is also considered, and numerical examples are included. The procedure can be adjusted satisfactorily whatever the arrangement of the roots of the "original" equation may be, and is suitable for use in a digital-computer programme. A significant saving of machine time relative to an iterative process dealing with the "new" equation in isolation is to be expected.

1. INTRODUCTION

In a recent investigation undertaken by the author concerning the propagation of electromagnetic waves through the ionosphere, it was necessary to solve a number of quartic equations having complex coefficients which varied slowly (and normally continuously) with height; horizontally the ionospheric conditions were assumed to be constant.

A computer programme (in Elliott autocode) was devised for solving such a quartic ab initio, and for deducing the relevant eigenvectors associated with ionospheric propagation at any height, but there is a possibility of saving considerable machine time if advantage is taken of the fact that the roots of the quartic vary continuously with height.

In straightforward cases where the roots of the quartic at the initial height Z_0 are well separated, the roots at a neighbouring height $Z_0 - h$ can be deduced explicitly but approximately by an adaptation of Newton's method of root approximation. This adaptation is explained in Section 2, and its limitations are discussed. Alternative procedures required when the roots are not well separated are considered in Section 3. Extension to equations of higher degree is briefly considered in Section 4, and Section 5 gives examples. There is thus no essential originality in this report; its objective is the purely practical one of making other people's work easier.

2. THE CASE OF WELL SEPARATED ROOTS

Suppose that the original equation is:

$$F(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$
$$= (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4) = 0 \qquad (1)$$

and that the roots λ_1, λ_2, λ_3 and λ_4 are well separated, so that none of the differences $\lambda_i - \lambda_j$ need be regarded as small. Now suppose that the coefficients a_i are changed to $a_i + \delta a_i$ where the changes δa_i can be regarded as small. Then the resulting changes $\delta\lambda_i$ in the roots are given (to the first order) by Newton's formula:

$$\delta\lambda_i = -\frac{F(\lambda_i) + \phi(\lambda_i)}{F^1(\lambda_i)} = -\frac{\phi(\lambda_i)}{F^1(\lambda_i)} \qquad (2)$$

where

$$\phi(x) = \delta a_3x^3 + \delta a_2x^2 + \delta a_1x + \delta a_0 \qquad (3)$$

and $F^1(x)$ means the derivative of $F(x)$ with respect to x. Equation (2) shows that it is the size of $\phi(\lambda_i)$ rather than that of δa_i that matters. Now it can be shown that:

$$F^1(\lambda_i) = \Pi(\lambda_i - \lambda_j) \qquad (4)$$

where the symbol Π indicates the product of all factors $(\lambda_i - \lambda_j)$ for which $j \neq i$. Equation (4) indicates that $F^1(\lambda_i)$ will be small if any of the other roots λ_j is close to λ_i, and explains the well-known fact that (2) is unsatisfactory in such cases. But if the numerator $\phi(\lambda_i)$ is also sufficiently small, (2) can still be used. Furthermore, $\phi(\lambda_i)$ and the original roots λ_i are known ab initio, so that when the original roots are well separated,

$F^1(\lambda_i)$ is not small and $\delta\lambda_i$ is of the same order of magnitude as $\phi(\lambda_i)$. If two roots of (1) are close together, so that $(\lambda_i - \lambda_k)$ is small but none of the other differences $(\lambda_i - \lambda_j)$ are, then $\phi(\lambda_i)$ must be of order $(\lambda_i - \lambda_k)^2$ to permit the use of (2) as it stands. If three roots are close together, so that $(\lambda_i - \lambda_k)$ and $(\lambda_i - \lambda_l)$ are both small quantities of the same order of magnitude, say M, then $F^1(\lambda_i)$ is of order M^2 from (4) and therefore $\phi(\lambda_i)$ must be of order M^3 to permit the use of (2).

When Equation (2) can be used, $(\lambda_i + \delta\lambda_i)$ is a closer approximation to a root of:

$$F(x) + \phi(x) = 0 \tag{5}$$

than λ_i was, and the procedure discussed above can be repeated so that the roots of (5) are obtained to any required degree of accuracy. For suppose we are seeking a better approximation to the root near $\lambda_1 + \delta\lambda_1$ and that:

$$\psi_1(x) = \frac{\phi(x)}{(x - \lambda_2)(x - \lambda_3)(x - \lambda_4)} \tag{6}$$

[so that $\psi_1(\lambda_1) = \phi(\lambda_1)/F^1(\lambda_1)$].

Then (5) can be written:

$$(x - \lambda_1) + \psi_1(x) = 0 \tag{7}$$

and Equation (2) (for $i = 1$) is equivalent to replacing $\psi_1(x)$ in (7) by $\psi_1(\lambda_i)$. If $\phi(x)$ is of the first order of small quantities, then so is $\psi_1(x)$ in the case of well-separated roots. We can therefore apply Taylor's theorem and write:

$$\psi_1(x) \approx \psi_1(\lambda_1) + (x - \lambda_1)\,\psi_1^1(\lambda_1) \tag{8}$$

Substituting from (8) into (7), we thus obtain:

$$x - \lambda_1 - \delta\lambda_1 = x - \lambda_1 + \psi_1(\lambda_1) = -(x - \lambda_1)\psi_1^1(\lambda_1) \tag{9}$$

If in (9) x is the root of (5) nearest λ_1, $(x - \lambda_1)$ is of the same order of magnitude as $\delta\lambda_1 = -\psi_1(\lambda_1)$ and therefore $(x - \lambda_1 - \delta\lambda_1)$ is of order $(\delta\lambda_1)^2$. Hence in the case of well-separated roots, the correction required to the approximation given by (2) is an order of magnitude smaller than the correction given by (2) itself to the initial approximation λ_1. (The argument is equally applicable to any other root λ_i, but the verbal expression of it is more complicated.)

In cases mentioned above where the roots are not well separated, but (2) can nevertheless still be used, the corresponding result can be shown to be that the correction given by (2) is of the same order of magnitude as the separations of the roots which are close together, while the correction subsequently required is of the order of the squares and products of the separations of the roots.

3. ALTERNATIVE PROCEDURE WHEN THE ROOTS ARE NOT WELL SEPARATED

The case in which the foregoing argument is unsatisfactory is that in which the roots λ_i are not all well separated, and where $\phi(\lambda_i)$, although small in the absolute sense, is not sufficiently small relative to $F^1(\lambda_i)$ to make $\delta\lambda_i$ small in Equation (2). Consider first the simplest case of this kind, when λ_1 and λ_2 are nearly equal but are well separated from λ_3 and λ_4. In such a case divide through (5) by $(x - \lambda_3)(x - \lambda_4)$ and let:

$$\psi_{12}(x) = \frac{\phi(x)}{(x - \lambda_3)(x - \lambda_4)} \tag{10}$$

Then instead of (7) we have:

$$(x - \lambda_1)(x - \lambda_2) + \psi_{12}(x) = 0 \tag{11}$$

which can be rearranged in the form:

$$\{x - \tfrac{1}{2}(\lambda_1 + \lambda_2)\}^2 = \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 - \psi_{12}(x) \tag{12}$$

Now in the case of well separated roots considered above, the first term on the right hand side of (12) is large compared to the second, whereas when $\lambda_1 \approx \lambda_2$ the two terms on the right hand side of (12) may be comparable or the term $\psi_{12}(x)$ may prevail. In any case we can replace $\psi_{12}(x)$ by $\psi_{12}\{\tfrac{1}{2}(\lambda_1 + \lambda_2)\}$ and then obtain the second approximations to the roots of (5) corresponding to λ_1 and λ_2 as:

$$M_1, M_2 = \tfrac{1}{2}(\lambda_1 + \lambda_2) \pm \left\{\left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 - \psi_{12}\left(\frac{\lambda_1 + \lambda_2}{2}\right)\right\}^{1/2} \tag{13}$$

If the term $\psi_{12}\{\tfrac{1}{2}(\lambda_1 + \lambda_2)\}$ prevails, the change in λ_1 and λ_2 is of order $[\psi_{12}\{\tfrac{1}{2}(\lambda_1 + \lambda_2)\}]^{1/2}$ (instead of having the same order as $\phi(\lambda_1)$ as was found in the well separated roots case).

A possible way of getting a better approximation to M_1 is to replace it by:

$$\mu_1 = \tfrac{1}{2}(\lambda_1 + \lambda_2) + \left\{\left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 - \psi_{12}(M_1)\right\}^{1/2} \tag{14}$$

and similarly to replace M_2 by:

$$\mu_2 = \tfrac{1}{2}(\lambda_1 + \lambda_2) - \left\{\left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 - \psi_{12}(M_2)\right\}^{1/2} \tag{15}$$

and this process has the same sort of convergence that is associated with repeated application of Newton's formula to the well separated roots case as explained in connection with Equation (9).

Alternatively, having obtained second approximations corresponding to (2) for the roots which are isolated and to (13) in the case of pairs of nearly equal roots, we can regard the equation (5) for which a solution is actually required as a variant of the equation whose roots are the best approximations hitherto obtained, instead of as a variant of the original equation (1). If this is done the procedure for well separated roots can normally be used thereafter. If m roots are clustered ($m > 2$) it may occasionally be necessary to use the clustered root procedure for n roots ($n < m$) subsequently, as in Example 3 below.

If the quartic equation (1) has two pairs of roots such that the members of each pair are close together, and each pair is well separated from the other pair, we can proceed as in the case of a single close pair. The procedure is affected by the separation or closeness of roots near the root being sought, but not by the separation or closeness of roots remote from the root being sought. If the quartic (1) has three roots close together and well separated from the remaining root (λ_4), (5) should be rewritten in the form:

$$(x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = -\frac{\phi(x)}{x - \lambda_4} = -\chi(x), \text{ say} \quad (16)$$

Now let:

$$x - u = y \qquad \lambda_1 = u + \eta_1$$
$$\lambda_2 = u + \eta_2 \qquad \lambda_3 = u + \eta_3 \quad (17)$$

where

$$u = \tfrac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3) \text{ so that } \eta_1 + \eta_2 + \eta_3 = 0$$

Then

$$(x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$$
$$= y^3 + (\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2)y - \eta_1\eta_2\eta_3 \quad (18)$$

If we substitute back into (16) and take all terms except y^3 or $(x - u)^3$ over to the right hand side, (16) becomes:

$$(x - u)^3$$
$$= \eta_1\eta_2\eta_3 - (\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2)(x - u) - \chi(x) \quad (19)$$
$$= \eta_1\eta_2\eta_3 - \chi(u) \text{ when } x = u$$

(19) thus gives us second approximations to the three roots, and the best procedure thereafter is to regard the given equation as a variant of the equation whose well-separated roots are these second approximations, instead as a variant of the original equation. Clusters of m roots ($m > 3$) can be treated similarly.

4. EXTENSION TO EQUATIONS OF HIGHER DEGREE

If the original equation corresponding to (1) has a higher degree n, but has well separated roots, Equation (2) is still valid, (except in so far as $\phi(x)$ now has n terms instead of four), and so is equation (4) except that there are now $(n - 1)$ factors instead of 3.

If the original equation has a cluster of k roots $\lambda_1, \lambda_2 \ldots \lambda_k$ which are close together and have average value u and actual values $(u + \eta_1)$, $(u + \eta_2)$ $\ldots$ $(u + \eta_k)$ where $\eta_1 + \eta_2 + \ldots + \eta_k = 0$, then the second approximation to the roots of this cluster is given by:

$$(x - u)^k = (-1)^{k+1}\eta_1\eta_2\eta_3\ldots\eta_k - \frac{\phi(u)}{(u - \lambda_{k+1})(u - \lambda_{k+2})\ldots(u - \lambda_n)}$$
$$= Z, \text{ say} \quad (20)$$

so that:

$$x - u = |Z|^{1/k} e^{j(\arg z + 2r\pi)/k} (r = 0, 1 \ldots k - 1)$$

When second approximations $\lambda_1', \lambda_2' \ldots\ldots \lambda_n'$ to all the roots have been found in this way, the given equation should be rewritten in the form:

$$(x - \lambda_1')(x - \lambda_2') \ldots (x - \lambda_n') + \Omega(x) = 0 \quad (21)$$

and thereafter it can usually be treated as an equation having well separated roots. An exception occurs in Example 3 below because we have at one stage a complex approximation to a root which is in fact real.

In the situation discussed above, we have had the advantage that an obvious good initial approximation to the roots of the equation to be solved is available. When an equation has to be solved ab initio, however, the first task is to find such an initial approximation. Various well known methods are available for this but when a whole system of equations with varying coefficients has to be solved, one member of the system can usually be found for which solution is specially easy.

5. NUMERICAL EXAMPLES

In the illustrative numerical examples which follow, both the "original" equation and the "new" equation with slightly different coefficients whose

roots are being sought are chosen to have exact roots which are stated initially, so that the manner in which the approximations derived approach the time values can be clearly seen. All numerical examples are quartics.

Example 1 (well separated roots)

Starting with the quartic equation:

$$(x - 0{\cdot}5)\,(x - 1)\,(x - 2)\,(x - 4)$$

$$= x^4 - 7{\cdot}5x^3 + 17{\cdot}5x^2 - 15x + 4 = 0 \qquad (22)$$

find the roots of:

$$x^4 - 7{\cdot}7x + 18{\cdot}45x^2 - 16{\cdot}263x + 4{\cdot}6494 = 0 \qquad (23)$$

of which the exact values are 0·6, 0·9, 2·1 and 4·1.

In the notation of Equations (1) and (3):

$$a_3 = -7{\cdot}5,\ a_2 = 17{\cdot}5,\ a_1 = -15,\ a_0 = 4$$

$$\delta a_3 = -0{\cdot}2,\ \delta a_2 = 0{\cdot}95,\ \delta a_1 = -1{\cdot}263,\ \delta a_0 = 0{\cdot}6494$$

$$\lambda_1 = 0{\cdot}5,\ \lambda_2 = 1,\ \lambda_3 = 2,\ \lambda_4 = 4 \qquad (24)$$

Equation (2) gives:

$$\left.\begin{aligned}
\delta\lambda_1 &= +\frac{0{\cdot}2304}{0{\cdot}5 \times 1{\cdot}5 \times 3{\cdot}5} = \frac{0{\cdot}2304}{2{\cdot}625} = 0{\cdot}0878\\
\delta\lambda_2 &= -\frac{0{\cdot}1364}{0{\cdot}5 \times 1 \times 3} = -0{\cdot}0909\\
\delta\lambda_3 &= +\frac{0{\cdot}3234}{1{\cdot}5 \times 1 \times 2} = 0{\cdot}1078\\
\delta\lambda_4 &= +\frac{2{\cdot}0026}{3{\cdot}5 \times 3 \times 2} = 0{\cdot}0954
\end{aligned}\right\} \qquad (25)$$

To make further progress, we regard (23) as a variant of:

$$(x - 0{\cdot}5878)\,(x - 0{\cdot}9091)\,(x - 2{\cdot}1078)\,(x - 4{\cdot}0954)$$

$$= x^4 - 7{\cdot}7001x^3 + 18{\cdot}452223x^2 - 16{\cdot}236464x + 4{\cdot}612825 = 0 \qquad (26)$$

instead of as a variant of (22) so that the revised values of δa_3 etc. become:

$$\delta a_3^1 = +0{\cdot}0001 \qquad \delta a_2^1 = -0{\cdot}002223$$

$$\delta a_1^1 = -0{\cdot}026536 \qquad \delta a_0^1 = 0{\cdot}036575 \qquad (27)$$

Hence:

$$\left.\begin{aligned}
\delta\lambda_1^1 &= +\frac{0{\cdot}020229}{0{\cdot}3213 \times 1{\cdot}52 \times 3{\cdot}5076} = \frac{0{\cdot}020229}{1{\cdot}713028}\\
&= 0{\cdot}01181\\
\delta\lambda_2^1 &= -\frac{0{\cdot}010689}{0{\cdot}3213 \times 1{\cdot}1987 \times 3{\cdot}1863} = -\frac{0{\cdot}010689}{1{\cdot}227179}\\
&= -0{\cdot}00871\\
\delta\lambda_3^1 &= -\frac{0{\cdot}028297}{1{\cdot}52 \times 1{\cdot}1987 \times 1{\cdot}9876} = -\frac{0{\cdot}028297}{3{\cdot}621455}\\
&= -0{\cdot}00781\\
\delta\lambda_4^1 &= +\frac{0{\cdot}102509}{3{\cdot}5076 \times 3{\cdot}1863 \times 1{\cdot}9876} = \frac{0{\cdot}102509}{22{\cdot}213944}\\
&= +0{\cdot}00461
\end{aligned}\right\} \qquad (28)$$

Hence we have all the correct roots with errors only in the fourth decimal place after two applications of the Newton process. In this illustrative example only real roots were involved at any stage. In principle no difference occurs when there are complex roots. For a desk-machine calculation, however, the work is more tedious, but this is immaterial if a digital computer is used; programming is reasonably straigthforward.

Example 2 (One pair of roots nearly equal; complex roots involved)

Starting with the quartic equation:

$$(x - 0{\cdot}9)\,(x - 1{\cdot}1)\,(x - 2)\,(x - 4)$$

$$= x^4 - 8x^3 + 20{\cdot}99x^2 - 21{\cdot}94x + 7{\cdot}92 = 0 \qquad (29)$$

find the roots of:

$$x^4 - 8{\cdot}1x^3 + 22{\cdot}1025x^2 - 24{\cdot}411x + 9{\cdot}5513 = 0 \qquad (30)$$

of which the exact values are $1{\cdot}05 \pm 0{\cdot}2j$, 2·2 and 3·8.

Equation (12) becomes:

$$(x - 1)^2 = 0{\cdot}01 - \frac{0{\cdot}1728}{3} = -0{\cdot}0476 \qquad (31)$$

so the second approximation for the roots near 1 is $1 \pm 0{\cdot}218174j$, while the second approximations to the other two roots are 2·1714 and 3·8250. Henceforward we therefore regard (30) as a variant of:

$$(x^2 - 2x + 1{\cdot}0476)\,(x - 2{\cdot}1714)\,(x - 3{\cdot}8250)$$

$$= x^4 - 7{\cdot}9964x^3 + 21{\cdot}3460x^2 - 22{\cdot}893029x + 8{\cdot}700947 = 0 \qquad (32)$$

and the changes in coefficients are thus reduced to:

$$\delta a_3^1 = -0{\cdot}1036; \qquad \delta a_2^1 = 0{\cdot}7565;$$

$$\delta a_1^1 = -1{\cdot}517971; \qquad \delta a_0^1 = 0{\cdot}850353; \qquad (33)$$

$\delta\lambda_3^1$ then is found to be 0·0281, which gives the next approximation to λ_3 as 2·1995, which is very close to the correct value 2·2. Likewise $\delta\lambda_4^1$ is found to be -0·0237, so that the second approximation to λ_4 is 3·8013, which again is very close to the correct value 3·8.

To obtain $\delta\lambda_1^1$, we can write (30) in the form:

$$x - 1 - 0{\cdot}218174\mathrm{j} = \frac{-0{\cdot}1036x^3 + 0{\cdot}7565x^2 - 1{\cdot}517971x + 0{\cdot}850353}{(x - 1 + 0{\cdot}218174\mathrm{j})\,(x - 2{\cdot}1714)\,(x - 3{\cdot}8250)} \qquad (34)$$

On the right hand side of (34) divide the numerator by $x^2 - 2x + 1{\cdot}0476$, and replace x by $1 + 0{\cdot}218174\mathrm{j}$ in the remainder and also in the factor $x - 1 + 0{\cdot}218174\mathrm{j}$ in the denominator. In the remaining (real) denominator factors we can either replace x by $1 + 0{\cdot}218174\mathrm{j}$, or preferably multiply these factors out, divide by $x^2 - 2x + 1{\cdot}0476$ and then replace x by $1 + 0{\cdot}218174\mathrm{j}$ in the linear remainder. In this way we find:

$$x - 1 - 0{\cdot}218174\mathrm{j} \approx 0{\cdot}0508 - 0{\cdot}0117\mathrm{j}$$

so that $\lambda_1 \approx 1{\cdot}0508 - 0{\cdot}2064\mathrm{j}$ and λ_2 is the conjugate of this. The real part of λ_1 is very close to the correct value; the error in the imaginary part is greater, but this is not surprising since approximations to this imaginary part are in effect obtained by taking the square root of a small difference of relatively large quantities.

<u>Example 3</u>

Starting with the quartic equation

$$(x - 1)^3(x - 3) = x^4 - 6x^3 + 12x^2 - 10x + 3 = 0 \qquad (35)$$

find the roots of

$$x^4 - 6{\cdot}15x^3 + 12{\cdot}7475x^2 - 11{\cdot}039625x + 3{\cdot}441375 = 0 \qquad (36)$$

the exact values of which are 0·95, 1·05, 1·15 and 3.

In the notation of equation (3):

$$\delta a_3 = -0{\cdot}15: \qquad \delta a_2 = 0{\cdot}7475;$$

$$\delta a_1 = -1{\cdot}039625; \qquad \delta a_0 = 0{\cdot}441375 \qquad (37)$$

and we then find immediately that the quantity called $\phi(x)$ in equation (3) is zero for $x = 3$, and therefore $\delta\lambda_4 = 0$ exactly. We prefer not to use any special technique, taking advantage of this, as our main object is illustration of a general method adaptable for programming; under such circumstances it is preferable to have as few special cases as possible, whereas a few extra repetitions of the normal routine are not particularly objectionable. However, the next step is to write (36) in the form:

$$(x - 1)^3 = -\frac{-0{\cdot}15x^3 + 0{\cdot}7475x^2 - 1{\cdot}039625x + 0{\cdot}441375}{x - 3}$$

$$= +0{\cdot}15x^2 - 0{\cdot}2975x + 0{\cdot}147125 \qquad (38)$$

$$= -0{\cdot}000375 \text{ when } x = 1$$

So the next approximations to λ_1, λ_2 and λ_3 are respectively:

$$x = 1 + 0{\cdot}072112e^{(2k+1)\pi\mathrm{j}/3} \quad (k = 0,1,2)$$

$$= 0{\cdot}927888,\ 1{\cdot}036056 \pm 0{\cdot}062451\mathrm{j}$$

We therefore next regard (36) as a variant of:

$$\{(x - 1)^3 + 0{\cdot}000375\}\,\{x - 3\} = 0 \qquad (39)$$

$$\text{or } x^4 - 6x^3 + 12x^2 - 9{\cdot}999625x + 2{\cdot}998875 = 0$$

so that δa_3, δa_2 are unaltered, $\delta a_1 = -1{\cdot}04$ and $\delta a_0 = 0{\cdot}4425$. Hence (36) can be written:

$$(x - 1) + 0{\cdot}072112 = \frac{+0{\cdot}15x^3 - 0{\cdot}7475x^2 + 1{\cdot}04x - 0{\cdot}4425}{(x - 3)\,\{(x - 1)^2 - 0{\cdot}072112\,(x - 1) + 0{\cdot}005200\}} \qquad (40)$$

and the next approximation to the real root is obtained by putting $x = 0{\cdot}927888$ on the right hand side, the numerator of which in this case happens to be $0{\cdot}15(x - 3)\,(x - 1)\,(x - 0{\cdot}983333)$. The right hand side of (40) is therefore:

$$\frac{0{\cdot}15 \times 0{\cdot}072112 \times 0{\cdot}055445}{0{\cdot}0156} = 0{\cdot}038445$$

and the corresponding approximation for a root of (36) is 0·9663. The corresponding equation to (40) when we seek to improve the approximation $1{\cdot}036056 + 0{\cdot}062451\mathrm{j}$ is:

$$x - 1{\cdot}036056 - 0{\cdot}062451\mathrm{j} = \frac{0{\cdot}15(x - 3)\,(x - 1)\,(x - 0{\cdot}983333)}{(x - 3)\,(x - 0{\cdot}927888)\,(x - 1{\cdot}036056 + 0{\cdot}062451\mathrm{j})} \qquad (41)$$

$$\therefore\ x - 1{\cdot}036056 - 0{\cdot}062451\mathrm{j}$$

$$= \frac{0{\cdot}15\,[(x^2 - 2{\cdot}072112x + 1{\cdot}077312) + (0{\cdot}088779x - 0{\cdot}093979)]}{2 \times 0{\cdot}062451\mathrm{j} \times (x - 0{\cdot}927888)} \qquad (42)$$

When $x = 1{\cdot}036056 + 0{\cdot}062451\mathrm{j}$ and $x^2 - 2{\cdot}072112x + 1{\cdot}077312 = 0$, (42) reduces to:

$$x - 1{\cdot}036056 - 0{\cdot}062451\mathrm{j} = 0{\cdot}055775 - 0{\cdot}01\mathrm{j}$$

so that $x = 1{\cdot}091831 + 0{\cdot}052451\mathrm{j}$, and this number and its conjugate are the required approximations. We therefore now regard (36) as a variant of:

$$(x - 0{\cdot}9663)\,(x^2 - 2{\cdot}183662x + 1{\cdot}194846)\,(x - 3) = 0 \qquad (43)$$

or

$$x^4 - 6{\cdot}149962x^3 + 12{\cdot}754805x^2 - 11{\cdot}069335x + 3{\cdot}463739 = 0$$

and it is now clear that we have approached the given equation (36) much more closely than was apparent by comparing the approximations found to the roots with the exact roots of (36). We now treat (36) as a variant of (43) using the procedure of Section 2 for well separated roots once more. We have, in the notation of (3):

$$\phi(x)$$

$$= -0{\cdot}000038x^3 - 0{\cdot}007305x^2 + 0{\cdot}029710x - 0{\cdot}022634$$

$$= -(x - 3)\,(0{\cdot}000038x^2 + 0{\cdot}007419x - 0{\cdot}007453) \qquad (44)$$

So the next approximation to the root near $0{\cdot}9663$ is given by:

$$x - 0{\cdot}9663 = \left[\frac{0{\cdot}000038x^2 + 0{\cdot}007419x - 0{\cdot}007453}{x^2 - 2{\cdot}183662x + 1{\cdot}194846}\right]_{x\,=\,0{\cdot}9663}$$

$$= \frac{-0{\cdot}00024854}{0{\cdot}018509} = -0{\cdot}01343 \qquad (45)$$

so that $x \approx 0{\cdot}9529$. The corresponding next approximation to the root near $1{\cdot}091831 - 0{\cdot}052451\mathrm{j}$ is found, by a calculation analogous to equations (41) and (42), to be $1{\cdot}0987 - 0{\cdot}0027\mathrm{j}$.

Now Equation (38) indicates that the initial approximation of three equal roots all unity is inevitably followed by one real root and two complex conjugates (for an equation having real coefficients) whereas in fact (36) has three real roots (apart from $x = 3$). The approximation $1{\cdot}0987 \pm 0{\cdot}0027\mathrm{j}$ obtained from (44) has a much smaller imaginary part than the previous approximation $1{\cdot}091813 \pm 0{\cdot}052451\mathrm{j}$, and this indicates that the next move is to regard (36) as a variant of:

$$(x - 1{\cdot}0987)^2\,(x - 0{\cdot}9529)\,(x - 3)$$

$$= x^4 - 6{\cdot}1503x^3 + 12{\cdot}751944x^2 - 11{\cdot}053419x + 3{\cdot}450857 = 0 \qquad (46)$$

If we now apply the procedure of Section 3 for an isolated pair of equal roots, we obtain as the next approximation:

$$(x - 1{\cdot}0987)^2$$

$$= -\frac{0{\cdot}0003x^3 - 0{\cdot}004444x^2 + 0{\cdot}013794x - 0{\cdot}009482}{(x - 0{\cdot}9529)\,(x - 3)}$$

$$= -\frac{0{\cdot}0003x^2 - 0{\cdot}003544x + 0{\cdot}003161}{x - 0{\cdot}9529}$$

$$= +0{\cdot}0025422 \text{ when } x = 1{\cdot}0987 \qquad (47)$$

indicating that the two roots in question are in fact real, and that the next approximations to them are $1{\cdot}1491$ and $1{\cdot}0483$ respectively, which are very close to the true values $1{\cdot}15$ and $1{\cdot}05$. The corresponding value for the root near $0{\cdot}9529$ is given by:

$$x - 0{\cdot}9529$$

$$= -\left[\frac{0{\cdot}0003x^2 - 0{\cdot}003544x + 0{\cdot}003161}{(x - 1{\cdot}0987)^2}\right]_{x\,=\,0{\cdot}9529}$$

$$= -\frac{0{\cdot}000056328}{0{\cdot}02125764} = -0{\cdot}00265 \qquad (48)$$

The next step would therefore be to regard the equation as a variant of:

$$(x - 0{\cdot}9503)\,(x - 1{\cdot}0483)\,(x - 1{\cdot}1491)\,(x - 3)$$

$$= x^4 - 6{\cdot}1477x^3 + 12{\cdot}735891x^2 - 11{\cdot}023107x + 3{\cdot}434200 = 0 \qquad (49)$$

and the accurate determination of the roots is now straightforward - the procedure of Section 2 for well separated roots only is required. It is at first sight surprising that the discrepancies between the coefficients of (49) and (36) are greater than those between the coefficients of (46) and (36), but it is the values of the quantity called $\phi(x)$ in Equation (3) when x is equal to an approximate root of (36) which control the situation.

6. ADAPTATION FOR A COMPUTER PROGRAMME

The above examples indicate that the procedure we have discussed should be satisfactory in all cases. It should also be straightforward for programming. The basic process involved is the determination of $\delta\lambda_i$ in Equation (2); this requires the determination of the ratio of two polynomials for an argument which may be complex; what is involved is therefore division of the polynomials by a linear complex divisor (or alternatively a quadratic divisor with real coefficients, and substitution of a complex number in a linear remainder). It is also necessary to test whether $|\lambda_i - \lambda_j|$ is ever small, and if so, how many roots are such that say:

$$|\lambda_i - \lambda_j| < 0{\cdot}1\ |\lambda_i + \lambda_j| \tag{50}$$

If there are n roots satisfying (50), then the procedure of Sections 3 and 4 must be invoked.

Even when this procedure does have to be invoked, what is needed is still the determination of the ratio of two polynomials for complex argument, and the nth root of this ratio must also be taken. If $n = 2$, the equation will not require this procedure more than once. If $n = 3$, as in Example 3, the special procedure may have to be invoked a second time for a lower value of n, usually 2, but not thereafter. This possibility, however, is automatically catered for by means of the test (50).

Subroutines for the evaluation of a polynomial for real or complex argument (with complex coefficients if necessary), and for the determination of the polynomial whose zeros have known values, will be required repeatedly.

The question remains of obtaining the roots of an initial equation which cannot be regarded as a variation of an equation already obtained. In the case of ionospheric propagation which stimulated the present investigation, it is known that at the bottom of the ionosphere, the relevant quartic equation has two pairs of equal roots, and therefore we can start calculating eigen-values at the bottom of the ionosphere and work upwards, even if we subsequently have to consider the behaviour of an electromagnetic wave incident at the top of the ionosphere and trace its progress downwards. Likewise in other cases where systems of equations have to be considered with varying coefficients, there may often be one individual member of the system for which the equation is easily soluble by a special technique.

7. CONCLUSIONS

If a "new" algebraic equation can be regarded as a variation of an "original" equation whose roots are already known, the "new" equation can be solved in terms of the "original" equation by an adaptation of Newton's approximation, whatever the relative positions of the roots of the "original" equation. The adaptation in question can be satisfactorily formulated as a computer programme.

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